Singapore International Mathematics Challenge 2014

Experior. Expono. Excedo.

THEME: LAND TRANSPORT

About the Challenge:

There are three parts to the Challenge – Parts A, B & C. Each part consists of 1 to 2 questions. The weighting for each question indicates its contribution to the final score. It also serves as a guide on the amount of time and effort your team should spend on the question.

Written Report Requirements:

1. Your report should not exceed 10 pages. It will be read by judges. Present the main results and ideas only. You may omit details such as the steps of computations. However, you should be able to explain the details of your work during the oral presentations, if requested by the judges.

2. Teams should acknowledge all sources used.

3. Your report must:
   - Be saved in PDF format.
   - Be single spacing with font size 12. Do not try to squeeze in the details by reducing the spacing between lines, or the margins.
   - Include your school name in FULL, on the first page of your report.

Oral Presentations:

1. Each team has 20 minutes to present their solutions to the judges. This will be followed by a 5-minute Question & Answer session.

2. Each team has to present 3 times – each presentation covers all three parts.

There are 10 printed pages (excluding the pink cover page).

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Part A: Tourist in a hurry

You are a tourist. You have a list of attractions that you are going to visit. You have set aside a certain amount of money for local transportation. There are three modes of transport – on foot (free but slow), public transport (cheap), and taxi (fast but expensive). Your goal is to figure out the optimal route: you always begin at your hotel and come back to it, but you can choose the order in which you visit the attractions and how you travel to the next attraction. You want to minimise the total time you spend travelling between attractions and stay within your transport budget.

Question 1 (20%)

You do **NOT** have to implement any of the algorithms as computer programs.

(a) **Exhaustive enumeration.** Suppose that you are given all the data: the budget, the list of attractions, and the tables of costs and times needed to travel between locations. One algorithm to find the optimal route and the associated transport modes is to enumerate all the possible routes and transport modes exhaustively, and select the one with the minimal total time that is within your transport budget. Given a list of *n* locations (attractions and hotel), what is the time complexity for this exhaustive algorithm to solve the problem? In other words, how many candidate solutions must be examined? [For information on the time complexity of an algorithm, refer to http://en.wikipedia.org/wiki/Time_complexity.]

(b) **More effective exact solver.** Propose another algorithm that can work out the optimal route and the associated transport modes, but is slightly faster than the naïve algorithm in **(a)**. State and explain your algorithm and state its time complexity.

(c) **Fast approximate solver.** What if there are many attractions? The algorithms in **(a)** and **(b)** are too ineffective (i.e. they take far too long to be of any practical use). You need a much more effective algorithm, i.e., one that runs fast and gives an answer that is close to the optimum. In particular, the algorithm must be effective enough that you can perform by hand, without implementing a computer program. Your algorithm does not need to find the optimal route in this case, but it should still find a reasonably good one (i.e., the total travelling time is close to the minimum, and the transport cost is within the budget constraint). State and explain your algorithm, justify your design (i.e. explain why your algorithm usually finds a reasonably good solution), state its time complexity and explain why it is effective.

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Apply your ‘fast approximate’ algorithm to the following data, stating the route and transport modes found by the algorithm. State the total transport time and cost.

- Budget for transportation – 20 dollars.
- Locations (Attractions and hotel):
  (1) Your hotel, Marina Bay Sands. You should begin and end your journey at the hotel.
  (2) Singapore Flyer
  (3) Vivo City
  (4) Resorts World Sentosa
  (5) Buddha Tooth Relic Temple at Chinatown
  (6) Singapore Zoo

Three tables of travelling times and costs for all the three modes of transport are as follows (Pages 3 - 5):

Travelling Time Required and Cost of Transportation

Travel by Public Transportation

To: / From:	Marina Bay Sands	Singapore Flyer	Vivo City	Resorts World Sentosa	Buddha Tooth Relic Temple	Zoo
Marina Bay Sands		$0.83 17 min	$1.18 26 min	$4.03 35 min	$0.88 19 min	$1.96 84 min
Singapore Flyer	$0.83 17 min		$1.26 31 min	$4.03 38 min	$0.98 24 min	$1.89 85 min
Vivo City	$1.18 24 min	$1.26 29 min		$2.00 10 min	$0.98 18 min	$1.99 85 min
Resorts World Sentosa	$1.18 33 min	$1.26 38 min	$0.00* 10 min		$0.98 27 min	$1.99 92 min
Buddha Tooth Relic Temple	$0.88 18 min	$0.98 23 min	$0.98 19 min	$3.98 28 min		$1.91 83 min
Zoo	$1.88 86 min	$1.96 87 min	$2.11 86 min	$4.99 96 min	$1.91 84 min	

* The public transportation from “Resorts World Sentosa” to “Vivo City” is free-of-charge.

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Travel by Taxi

To: / From:	Marina Bay Sands	Singapore Flyer	Vivo City	Resorts World Sentosa	Buddha Tooth Relic Temple	Zoo
Marina Bay Sands		$3.22 3 min	$6.96 14 min	$8.50 19 min	$4.98 8 min	$18.40 30 min
Singapore Flyer	$4.32 6 min		$7.84 13 min	$9.38 18 min	$4.76 8 min	$18.18 29 min
Vivo City	$8.30 12 min	$7.96 14 min		$4.54 9 min	$6.42 11 min	$22.58 31 min
Resorts World Sentosa	$8.74 13 min	$8.40 14 min	$3.22 4 min		$6.64 12 min	$22.80 32 min
Buddha Tooth Relic Temple	$5.32 7 min	$4.76 8 min	$4.98 9 min	$6.52 14 min		$18.40 30 min
Zoo	$22.48 32 min	$19.40 29 min	$21.48 32 min	$23.68 36 min	$21.60 30 min	

Travel on Foot

To: From:	Marina Bay Sands	Singapore Flyer	Vivo City	Resorts World Sentosa	Buddha Tooth Relic Temple	Zoo
Marina Bay Sands		14 min	69 min	76 min	28 min	269 min
Singapore Flyer	14 min		81 min	88 min	39 min	264 min
Vivo City	69 min	81 min		12 min	47 min	270 min
Resorts World Sentosa	76 min	88 min	12 min		55 min	285 min
Buddha Tooth Relic Temple	28 min	39 min	47 min	55 min		264 min
Zoo	269 min	264 min	270 min	285 min	264 min	

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Part B: Bus stops

Bus is the most popular type of transport in Singapore, averaging over 3.3 million passenger-trips daily. Therefore, planning new bus routes and optimising existing ones are of top priority. Each individual bus route can usually be represented by a line of a fixed length with a certain number of bus stops located between its start and end. The commuters want the bus stops located so as to minimise their travel time. So the quantity to be minimised is the average journey time of all the commuters.

The model is as follows. There is a street of length say *L* km. The commuter population is distributed uniformly along the street. We need to find the number of bus stops and the optimal positions for them to minimise the average time a commuter spends on the way from a random point along the street to another random point. In order to get from a point P to a point Q, a commuter always walks to the bus stop nearest to P, then takes a bus ride to the bus stop nearest to Q, and then walks to Q. If two bus stops are at the same distance from P, then the commuter chooses either one randomly (and similarly if two bus stops are the same distance from Q). The walking speed is *W* km/h, the speed of a bus is *B* km/h, and a bus spends an additional *S* hours per stop to load and unload passengers. We shall use the notation *T*(P,Q) for the time the commuter spends travelling from P to Q.

Example

Consider the following street layout:

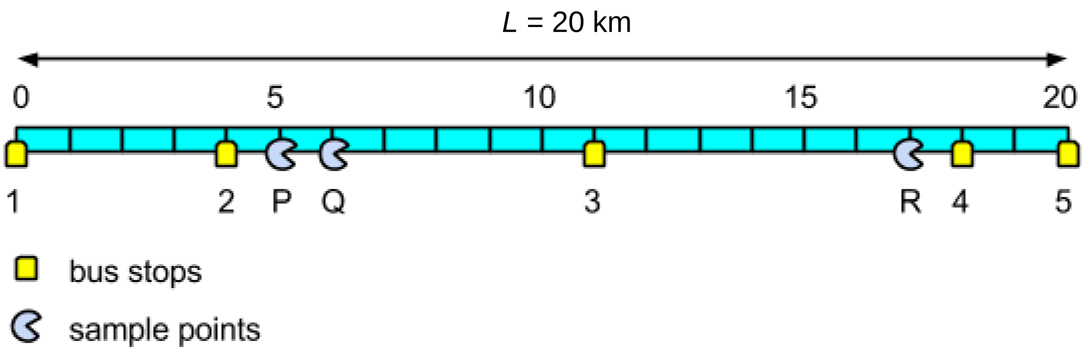

Figure 1

There are five bus stops and 3 sample points here. In this example, all distances between them are whole numbers. In general, the distances might be any real numbers. Let us find some timings:

- To get from P to R, the commuter walks 1 km to bus stop 2, then travels two stops and 14 km by bus to bus stop 4, then walks 1 km to R. We have then $T(\mathrm{P},\mathrm{R}) = \frac{1}{W} + \frac{14}{B} + 2\cdot S + \frac{1}{W} = \frac{2}{W} + \frac{14}{B} + 2S.$

- $T(\mathrm{Q},\mathrm{R}) = T(\mathrm{P},\mathrm{R}) + \frac{1}{W} = \frac{3}{W} + \frac{14}{B} + 2S.$
- To get from P to Q, the commuter walks 1 km to bus stop 2 because it is the nearest bus stop to P, then travels by bus to the same bus stop 2 which means he does not do anything, then walks to Q: $T(\mathrm{P},\mathrm{Q}) = \frac{1}{W} + \frac{0}{B} + 0 \cdot S + \frac{2}{W} = \frac{3}{W}.$ (Note: This is clearly not realistic.)

Let us fix some notations. There is always a bus stop at either end of the street. Let the positions of the bus stops be $0 = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = L.$. We write A for the complete arrangement of bus stops. Thus A may be described either by the vector $(x_1, x_2, \cdots, x_{n-1})$ or, equivalently by the vector $(d_1, d_2, \cdots, d_n)$, where $d_i = x_i - x_{i-1}$ for all $i = 1, 2, \ldots, n$.

Let E(A) be the expected (i.e. average) time of travel from a random point to another random point if the arrangement A of bus stops is fixed.

Question 2 (15%)

(a) Let n be fixed. Neglecting the time needed for a bus to drop and pick passengers (i.e., assuming that $S = 0$), prove that the optimal arrangement of the bus stops is when they are equidistant from each other – that is, the function E(A) attains its minimum value when $d_1 = d_2 = \cdots = d_n$ (or, equivalently, when $x_i = \frac{iL}{n}$ for all i). You may assume that such an optimal arrangement actually exists.

(b) Now let $L = 20, W = 5, B = 20, S = 0.05$. Find the value of n that minimises E(A), where A is the equidistant arrangement with n+1 stops and n intervals between them. (Since S is now non-zero, there is no assertion that the equidistant arrangement is the optimal one for a given value of n.)

Question 3 (25%)

The model proposed here has limitations. For instance, it does not take into account the fact that people can walk directly to the destination, that commuters go to certain places (like shops) more often than to other places and the commuter distribution is not uniform etc. Suggest ways to enrich the model to make it closer to reality and explain how similar problems like (a) and (b) could be solved in the new models.

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Part C: Traffic junction

Question 4 (15%)

(a) Consider a traffic junction as shown in Figure 2. Car C, which is travelling in the vertical direction, has come to a stop at the junction. It gives way to cars that are travelling in the horizontal direction. The problem is to determine the smallest possible spacing d between the cars travelling on the horizontal road such that it will be possible for car C to cross the junction without slowing down any of the cars on the horizontal road.

The assumptions are:

(i) The spacing between consecutive cars on the horizontal road is d.

(ii) The condition for no slowing down is that the centres of any two cars must be at distance at least s. (So in particular we must have $d \geq s$.)

(iii) All cars on the horizontal road travel from left to right at speed v, and Car C also travels at speed v. When Car C starts to move it accelerates instantly to speed v.

(iv) All cars travel in a straight line, either horizontally or, for Car C, vertically. No car changes direction.

Express your answer in terms of the variables s and/or v.

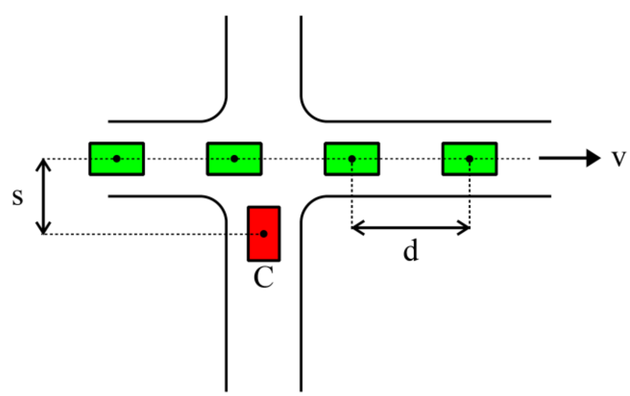

Figure 2

If you are a road planner, who is tasked to decide if this junction requires a traffic light, how may the information computed above be used?

(b) The assumptions made in **(a)** are of course over-simplified. In reality, the cars on the horizontal road will be arriving not at regularly-spaced times but at random times. To model this, we assume that the arrival rate at the junction of cars (on the horizontal road) is what is called a 'Poisson process of rate λ'.

What this means is that, in a very short time-interval δ, the chance that a car arrives is about $\lambda\delta$. It follows from this (you can just believe this or else you can look it up) that, in a time interval of length τ, the number of cars X that arrive during this interval satisfies

$$\mathrm{P}(X = k) = \frac{e^{-\lambda\tau}(\lambda\tau)^k}{k!} \qquad (1)$$

for each $k = 0,1,2,\ldots$. It also follows that, once a car has appeared, the waiting time Y until the next car appears satisfies

$$\mathrm{P}(Y \le r) = 1 - e^{-\lambda r} \qquad (2)$$

for each $r \ge 0$.

What is the expected (i.e. average) waiting time for Car C before it can cross the junction? If you need to make extra assumptions about the traffic flow then state them clearly.

(c) Now suppose that, as well as the number of cars arriving on the horizontal road being a Poisson process of rate λ, the number of cars arriving on the vertical road is also a Poisson process, of rate λ_v.

As before, each car on the vertical road must wait until there is a suitable break in the horizontal traffic before crossing. On average, how many cars (on the vertical road) are queueing at the junction?

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Question 5 (25%)

Actual traffic junctions are rarely uni-directional as they were in Question 4. Let us now consider a junction where cars travel in opposite directions along each branch, as shown in Figure 3.

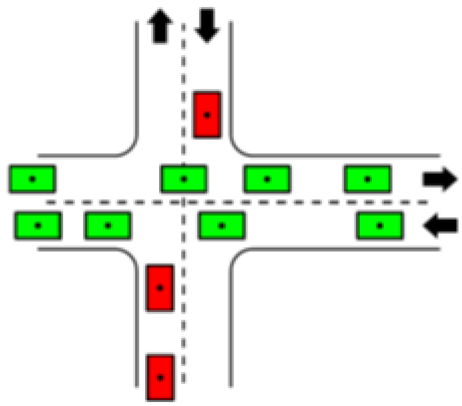
Figure 3

In this case, it often becomes virtually impossible for a waiting car to move off. Therefore, traffic lights are required to regulate the flow of traffic along the two roads. Formulate a model to design the timings of the traffic lights, such that the average queue length and waiting time along both roads are minimised. Feel free to propose any other criteria for the optimisation of a traffic junction.

(Note: You may assume more realistic conditions for the traffic flow. For example, the flow rate, vehicle density and other factors may vary according to the time of the day, etc. Any such considerations may result in bonus marks.)

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